

# Integration: Inverse trigonometric substitution

## Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

# Overview

- Inverse trigonometric substitution
- Other substitutions
  - Higher order roots
  - $\tan \frac{\theta}{2}$ -substitution

# Inverse trigonometric substitution

## What is inverse trigonometric substitution?

- Direct substitution: replace part of the integrand by a new variable
- Inverse substitution: **replace variable by new function.**

Example:  $\int \sqrt{1-x^2} dx$  with the substitution  $x = \cos \theta$

## Why is this useful?

Inverse trigonometric substitution can solve integrals that have a power of  $x^2 \pm a^2$  somewhere in the integrand.

Example:

- $\int \frac{dx}{(x^2 - 25)^3}$
- $\int \sqrt{1 + x^2} dx$

## Types of substitutions

The exact substitution depends on the integrand. We will distinguish the following three cases, depending on factors that occur in the integrand.

- 1 Factor  $\sqrt{a^2 - x^2}$ : use  $x = a \sin \theta$
- 2 Factor  $\sqrt{a^2 + x^2}$ : use  $x = a \tan \theta$
- 3 Factor  $\sqrt{x^2 - a^2}$ : use  $x = a \sec \theta$

Don't memorize these substitutions: use your insight into trigonometry to think what substitution would simplify the integrand.

## Remark: completing the square

Often the integrand will not exactly be in one of the three previous forms. In that case, **completing the square** may help:

$$\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x + 2)^2 + 1} = \int \frac{dt}{t^2 + 1} = \dots$$

## Case 1: $\sqrt{a^2 - x^2}$

- Substitution:  $x = a \sin \theta$ , with  $\theta \in [-\pi/2, \pi/2]$
- Differential:  $dx = a \cos \theta d\theta$

Example:  $\int \frac{dx}{(5 - x^2)^{3/2}}$



## Case 2: $\sqrt{a^2 + x^2}$

- Substitution:  $x = a \tan \theta$ , with  $\theta \in (-\pi/2, \pi/2)$
- Differential:  $dx = a \sec^2 \theta d\theta$

Example:  $\int \frac{dx}{\sqrt{1 + 4x^2}}$

### Case 3: $\sqrt{x^2 - a^2}$

- Substitution:  $x = a \sec \theta$ , with  $\theta \in (0, \pi)$  (**two cases**)
- Differential:  $dx = a \tan \theta \sec \theta d\theta$

Example:  $\int \frac{dx}{\sqrt{x^2 - a^2}}$

## Other substitutions

## Higher roots

Useful when the integrand has a factor  $\sqrt[n]{ax + b}$ .

- Substitution:  $ax + b = u^n$
- Differential:  $a dx = nu^{n-1} du$

Example:  $\int_{-1/3}^2 \frac{x dx}{\sqrt[3]{3x + 2}}$

## Example

Sometimes this rule can be applied to integrands that are somewhat more general than what's shown on the previous slide.

Example:  $\int \frac{dx}{x^{1/2}(1+x^{1/3})} = \dots$

(Substitute  $x^{1/6} = u$ )

## $\tan \frac{\theta}{2}$ -substitution

Useful when the integrand is a rational function of  $\sin \theta$  and  $\cos \theta$ .

- Substitution:  $t = \tan \frac{\theta}{2}$ , so that  $\theta = 2 \tan^{-1}(t)$
- Differential:

$$d\theta = \frac{2dt}{1+t^2}$$

- $t$ -formulas:

$$\sin \theta = \frac{2t}{1+t^2} \quad \text{and} \quad \cos \theta = \frac{1-t^2}{1+t^2}.$$

## Example

$$\int \frac{d\theta}{2 + \cos \theta} = \dots$$