

The binomial theorem

Introduction to Engineering Mathematics

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Overview

- Pascal's triangle
- Binomial coefficients
- Binomial theorem

Pascal's triangle

Expand the following expressions and look at the coefficients.

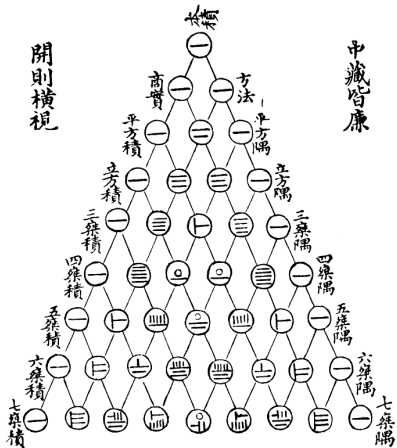
- $(a + b)^0 = 1$
- $(a + b)^1 = a + b$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

What do you notice?

Based on this pattern, what is $(a + b)^7$?

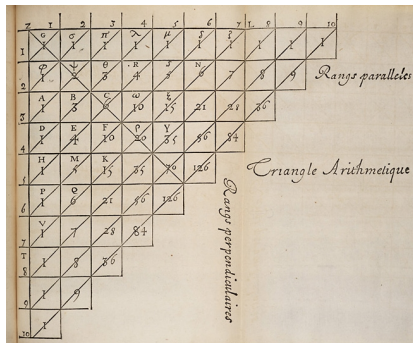
Would you be able to write down $(a + b)^{27}$?

古法七乘方圖



本積	方法	上廉	二廉	三廉	四廉	五廉	六廉	七廉
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Jian Xian (), 11th century CE



Blaise Pascal, 1665 CE

Example

Use Pascal's triangle to expand $\left(2x + \frac{1}{x}\right)^5$.

Binomial coefficients

- Factorial: $n! = n(n - 1)(n - 2) \cdots 2 \cdot 1$.
- Binomial coefficient (also called “n-choose-k”):

$$\binom{n}{k} = C_n^k = \frac{n!}{k!(n - k)!}.$$

- Measures the number of ways of choosing k objects from among n choices.

Properties

For all n and $k \leq n$:

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Rewriting Pascal's triangle using binomial coefficients

The binomial expansion

Putting everything we've learned together, we get

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

This can be written more compactly as

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$


Example

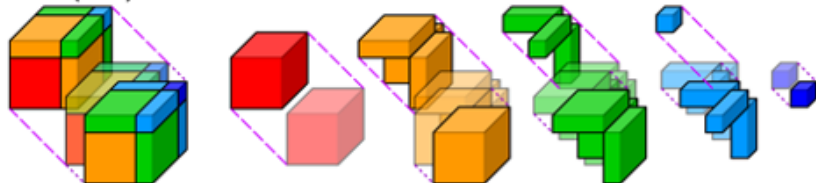
Use the binomial expansion to expand $(\sqrt{x} - 1)^7$.

Visual proof of the binomial expansion (optional)

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$
A large square is divided into four regions: a red square (a^2), a green L-shaped region (2ab), and a blue square (b^2). The L-shaped region is composed of two smaller green rectangles.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
A large cube is divided into four regions: a red cube (a^3), three yellow rectangular prisms (3a^2b), three green rectangular prisms (3ab^2), and a blue cube (b^3). The yellow prisms are arranged in a 2x2 grid with one missing, and the green prisms are arranged in a 2x2 grid with one missing.

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
A large cube is divided into five regions: a red cube (a^4), four orange rectangular prisms (4a^3b), six green rectangular prisms (6a^2b^2), four blue rectangular prisms (4ab^3), and a blue cube (b^4). The orange prisms are arranged in a 2x2 grid with one missing, the green prisms are arranged in a 2x2 grid with one missing, and the blue prisms are arranged in a 2x2 grid with one missing.